# Costly power and symbol rate allocation to sub-channels for optimal real performance: Water-filling for maximal throughput

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***Abstract*—We *analytically* allocate to sub-channels costly power and symbol rate to maximise a *general* throughput function. Link parameters (e.g., modulation order and packet length) are explicitly considered. We provide the structure of the general solution, and for certain scenarios give an explicit solution, which includes the optimal link configuration. A channel can be immediately discarded if its normalised gain is less than the normalised power price. Moreover, when power is "low", all of it should go to the best sub-channel. In general, the best "M" channels operate at maximal symbol rate, with the SNR that satisfies a single-variable optimising condition; and at most one additional channel operate at the "low power" optimum. Previously, we had analysed costly power allocation under a channel performance function that generalises the Gaussian capacity formula. However, the power allocation that would be optimal if each sub-channel operated at "capacity" could differ significantly from the maximiser of actual performance. Hence, the power allocation obtained herein is preferable.**

## I. Introduction

The sub-channel power allocation problem and its "water-filling" solution are well-known, when the performance function is logarithmic ("Gaussian capacity") and power is constrained but costless (for example, see [1, Sec. 5.3.3]). However, when power is costly, the problem seems much less explored. Until recently, Theorem 1 from [2] was the most relevant contribution of which we are aware, but it only addresses a specific logarithmic performance function, and it is stated without proof. Recently, [3] generalises this problem by considering both a per-Watt price, and a general concave performance function, which generalises the Gaussian capacity formula (desirable because the concerned sub-channels need not be Gaussian).

Now, we follow [3], but instead of sub-channel "capacity", we focus on actual throughput. As discussed below, throughput is not concave, but rather a product of power times a "bell shaped" (quasi-concave) function of the SNR. The present analysis comes closer to reality because actual performance can be significantly inferior to capacity. Thus, the power allocation that *would* be optimal if each sub-channel operated at "capacity", could be significantly different from the power allocation that maximises actual performance.

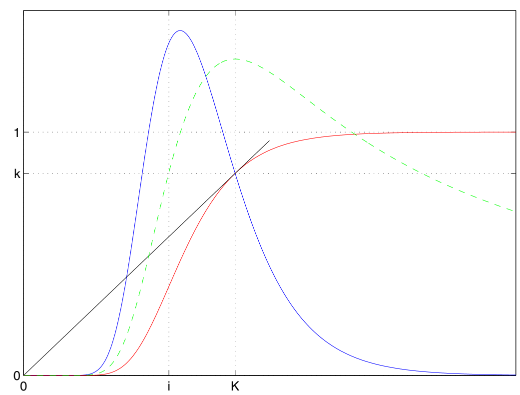


Figure 1: An S-curve, $S(x)$, $xS'(x)$ (solid bell curve) — the graph of $S'(x)$ is similar to that of $xS'(x)$) —, and the tangenu. $S(x)/x$ (dash, scaled) is maximised at $x = K$.

Additionally, here we include elements from [4], which — as suggested by [5] — characterises the combination of link parameters (power, symbol rate, modulation order, packet length, etc), that maximises "net" bits per second, or per Joule. To a *limited* extend, the present work extends [4] to a multi-channel scenario.

Below we first describe the system model. Then, we formulate the optimisation problem over symbol-rate and power, and immediately re-state it — for analytical convenience — in terms of power and SNR with appropriate constraints. Subsequently, we derive some simple, but non-trivial results from "first principles". Then, we undertake the Karush-Kuhn-Tucker (KKT) analysis [6], [7]. We end with a general discussion of our results.

## II. System model

### A. System model

- $M$ : no. of sub-channels ($m$ identifies a specific one)
- $c_0$ : unit price of power, and $b_0$ the value of one transferred information bit.
- $N_0$ : average Gaussian noise spectral density

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- $\hat{P}$ : the total power constraint
- $H_m > 0$ : channel gain with $H_1 \geq \cdots \geq H_M > 0$
- $R_m \leq \hat{R}_m$ :s ymbol rate
- $L_m$ : bits per packet, with $L_m - C_m$ information bits
- $b_m$ : bits per symbol, and $\bar{b}_m := b_m(L_m - C_m)/L_m$
- $p_m \geq 0$ : power allocated to sub-channel $m$
- $\sigma_m$ : per-symbol signal-to-noise ratio (SNR) :

$$\sigma_m = H_m p_m / (N_0 R_m) \tag{1}$$

- $F(x;\mathbf{a})$ is the packet-success rate function (PSRF) as a function of the *per-symbol* SNR $x$, given the combination of link parameters, $\mathbf{a}$. ($x$ may be used as a generic function argument). $f(x;\mathbf{a}) := F(x;\mathbf{a}) - F(0;\mathbf{a})$ replaces $F$ for some technical reasons[8]. For any $\mathbf{a}$, $f$ satisfies Definition A.1 (its graph as function of the *per symbol* SNR has the S-shape shown in Figs. 1 and 2.)
- $\mathcal{A} := \{\mathbf{a_1}, \cdots, \mathbf{a_K}\}$ : available link configurations

## III. Problem Statement and Re-formulation

### A. Information transferred over a period of interest

The number of information bits transferred over $\tau$ secs. with packet-success rate $f(x;\mathbf{a})$ and symbol rate $R \leq \hat{R}$ is given by

$$\tau \frac{L-C}{L} bRf(x;\mathbf{a}) \tag{2}$$

With $\tau = 1$, (2) yields the "goodput".

### B. Initial formulation

Let us maximise the value of the "goodput" minus its cost:

$$\max_{\substack{p_1,\cdots,p_m \\ R_1,\cdots,R_m}} b_0 \sum_{m=1}^{M} \bar{b}_m R_m f\left(\frac{H_m p_m}{N_0 R_m};\mathbf{a_m}\right) - c_0 \sum_{m=1}^{M} p_m \tag{3a}$$

subject to (3b)

$$\sum_{m=1}^{M} p_m \leq \hat{P} \tag{3c}$$

$$p_m \geq 0 \tag{3d}$$

$$0 \leq R_m \leq \hat{R}_m \tag{3e}$$

### C. Interesting quantities and functions

**Proposition III.1.** *If $x_m > 0$ and $(H_m/N_0)p_m/x_m \leq \hat{R}_m$, the contribution to throughput of sub-channel $m$ can be written as*

$$\frac{H_m p_m}{N_0} \frac{\bar{b}_m f(x_m;\mathbf{a_m})}{x_m} \tag{4}$$

*Proof:* $R_m \bar{b}_m f(H_m p_m/(N_0 R_m);\mathbf{a_m}) \equiv$

$$\frac{H_m p_m}{N_0} \frac{\bar{b}_m f(H_m p_m/(N_0 R_m);\mathbf{a_m})}{H_m p_m/(N_0 R_m)} \equiv \frac{H_m p_m}{N_0} \frac{\bar{b}_m f(x_m;\mathbf{a_m})}{x_m} \tag{5}$$

■

**Lemma III.1.** *Let $S : \Re^+ \to [0,d]$ satisfy Definition A.1 with inflexion at $z_f$. Let $\mathcal{B}(z) := S(z)/z$ with $\mathcal{B}(0) := \lim_{z\downarrow 0} \mathcal{B}(z) \equiv S'(0)$. (i) There is a unique tangent line from the origin to $S(z)$, denoted as $c^*z$ and called the tangenu, with tangency point, genu, $(z^*, S(z^*))$, where $z^* > z_f$. (ii) $\mathcal{B}$ is strictly quasi-concave, and its unique maximiser in $[0,Z]$ is $\min(z^*,Z)$ .*

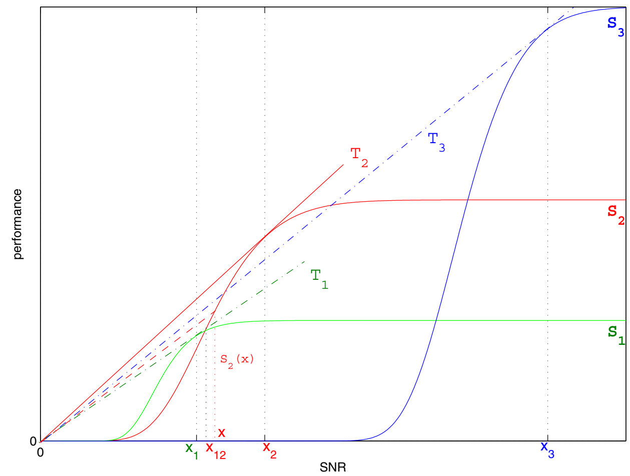



Figure 2: The combination of parameters yielding $S_2$ is best. Configuration 3 is eliminated. $S_1$ is used if $x_{12}$ (intercept of $T_1$ and $S_2$) cannot be reached (see [4]).

*Proof:* See [9]. ■

*Remark* III.1. Fig. 1 shows the tangenu and genu for $S(x) = [1 - \exp(-x/2)/2]^{80} - 2^{-80}$, and the graphs $xS'(x)$ and $S(x)/x$.

**Definition III.1.** Let $\mathbf{a}^*$ be such that $\forall \mathbf{a_k} \in \mathcal{A}$,

$$\rho^* := \bar{b}^* f(x^*;\mathbf{a}^*)/x^* \geq \bar{b}_k f(x_k^*;\mathbf{a_k})/x_k^* \tag{6}$$

*Remark* III.2. The "scaled" PSRF $\bar{b}^* f(\cdot;\mathbf{a}^*)$ has the "steepest" tangenu, among all those available. See Fig. 2 (from [4]).

Let the normalised channel gain be defined by

$$h_m := \rho^* \frac{H_m}{N_0} \implies \frac{H_m}{N_0} \equiv \frac{h_m}{\rho^*} \tag{7}$$

and

$$B(x_m;\mathbf{a_m}) := \frac{1}{\rho^*} \frac{\bar{b}_m f(x_m;\mathbf{a_m})}{x_m} \tag{8}$$

Then, (4) is identical to

$$\frac{\rho^* H_m}{N_0} p_m \frac{\bar{b}_m f(x_m;\mathbf{a_m})/x_m}{\rho^*} \equiv h_m p_m B(x_m;\mathbf{a_m}) \tag{9}$$

### D. Problem re-formulation

Let $c := c_0/b_0$. Problem (3) is restated as:

$$\max_{\substack{p_1,\cdots,p_m \\ x_1,\cdots,x_m}} \sum_{m=1}^{M} h_m p_m B(x_m;\mathbf{a_m}) - c \sum_{m=1}^{M} p_m \tag{10a}$$

subject to:

$$\sum_{m=1}^{M} p_m \leq \hat{P} \tag{10b}$$

$$h_m p_m - \rho^* \hat{R}_m x_m \leq 0 \tag{10c}$$

$$p_m \geq 0 \tag{10d}$$

$$x_m \geq 0 \tag{10e}$$

*Remark* III.3. By (7), (10c) is the symbol rate constraint:

$$R_m = H_m p_m/(N_0 x_m) \equiv h_m p_m/(\rho^* x_m) \leq \hat{R}_m$$

## IV. Basic results from first principles

**Proposition IV.1.** *If $h_n < c$ then $p_n = 0$ is optimal.*

*Proof:* By definition, (8), $B(x_n;\mathbf{a_n}) \leq 1$. $\therefore$ $h_n p_n B(x_n;\mathbf{a_n}) - c p_n \leq h_n p_n - c p_n \equiv p_n(h_n - c)$. ■

*Remark* IV.1. Without loss of generality, we assume that below $h_1 > \cdots > h_M > c$ ("useless" channels have been discarded).

**Proposition IV.2.** *Suppose $\mathbf{p} = (p_1, \cdots, p_M)$ and $\mathbf{x} = (x_1, \cdots, x_M)$ satisfy (10b)—(10e). If $x_m = x_n = x^*$ with $\mathbf{a_m} = \mathbf{a_n} = \mathbf{a}^*$ and $p_m h_m / x_m < \rho^* \hat{R}_m$ and $p_n h_n / x_n < \rho^* \hat{R}_n$, with $h_m > h_n$, then the pair $\mathbf{p}$, $\mathbf{x}$ does* not *solve (10).*

*Proof:*

The positive contribution of channels $m$ and $n$ to (10a) is $h_m p_m B(x^*;\mathbf{a}^*) + h_n p_n B(x^*;\mathbf{a}^*) \equiv h_m p_m + h_n p_n$.

Since $R_m \equiv p_m h_m / x^* < \rho^* \hat{R}_m$ then there is an $\varepsilon$ such that $(p_m + \varepsilon) h_m / x^* \leq \rho^* \hat{R}_m$ (this increases $p_m$ slightly and keep $x_m = x^*$, by raising $R_m$ without exceeding $\hat{R}_m$).

Replace $p_n$ with $p_n - \varepsilon$ and $p_m$ with $p_m + \varepsilon$

The net change to the objective function is $h_m \varepsilon - h_n \varepsilon$ which is positive, because $h_m > h_n$.

■

*Remark* IV.2. By Proposition IV.2, at the optimum, *at most* one channel will operate with a symbol rate below the constraint.

**Definition IV.1.** The communication system of (10) is said to be *under-powered* if $\hat{P} \times \max(H_1/\hat{R}_1, \cdots, H_M/\hat{R}_M) \leq N_0 x^*$.

*Remark* IV.3. Suppose that $\hat{R}_m = \hat{R} \quad \forall m$ (common symbol rate constraint). When $H_1 \hat{P}/x^* < N_0 \hat{R}$, the system cannot achieve the ideal SNR while operating at the highest available symbol rate even with all the power assigned to the best channel and using the best available link configuration.

**Proposition IV.3.** *If $\hat{P} \times \max(H_1/\hat{R}_1, \cdots, H_M/\hat{R}_M) \leq N_0 x^*$ then $p_1 = \hat{P}$, $x_1 = x^*$, $\mathbf{a_1} = \mathbf{a}^*$is optimal.*

*Proof:*

Under this hypothesis, the objective function (10a) yields $h_1 \hat{P} - c\hat{P}$.

For $0 < \Delta < \hat{P}$, set $p_1 = \hat{P} - \Delta$ and $p_m = \hat{P} - \Delta$.

It is now feasible and optimal to set $\mathbf{a_1} = \mathbf{a_m} = \mathbf{a}^*$ and $x_1 = x_m = x^*$.

Then, the new value of (10a) is $h_1(\hat{P} - \Delta) + h_m \Delta - c\hat{P}$; which yields a net change of $(-h_1 + h_m)\Delta$. This is negative, since $h_1 > h_2 > \cdots > h_M$ (by convention).

■

## V. KKT conditions and solutions

**Proposition V.1.** *If the vectors $(x_1, \cdots, x_M)$ and $(p_1, \cdots, p_M)$ form a (local) optimiser pair for Problem (10), then there are non-negative real numbers $\lambda_0, \lambda_1, \cdots, \lambda_M, \mu_1, \cdots .\mu_M, \nu_1 \cdots, \nu_M$ such that*

$$h_m\left(B(x_m;\mathbf{a_m}) - \mu_m\right) = c + \lambda_0 - \lambda_m \quad \forall m \tag{11a}$$

$$h_m p_m B'(x_m;\mathbf{a_m}) + \rho^* \hat{R}_m \mu_m = -\nu_m \quad \forall m \tag{11b}$$

$$\lambda_0 \left(P - \sum_{m=1}^{M} p_m\right) = 0 \tag{11c}$$

$$\mu_m\left(h_m p_m - \rho^* \hat{R}_m x_m\right) = 0 \tag{11d}$$

$$\lambda_m p_m = 0 \tag{11e}$$

$$\nu_m x_m = 0 \tag{11f}$$

*Proof:* These are the KKT conditions [6], [7]. ■

**Proposition V.2.** *If $p_m > 0$, then conditions (11a) and (11b) can, respectively, be re-written as*

$$h_m\left(B(x_m;\mathbf{a_m}) - \mu_m\right) = c + \lambda_0 \tag{12a}$$

$$h_m p_m B'(x_m;\mathbf{a_m}) + \rho^* \hat{R}_m \mu_m = 0 \tag{12b}$$

*which can be combined as*

$$h_m\left(B(x_m;\mathbf{a_m}) + \frac{h_m p_m B'(x_m;\mathbf{a_m})}{\rho^* \hat{R}_m}\right) = c + \lambda_0 \tag{12c}$$

*Proof:* (12a) and (12b) are implied by (11e) & (11f). Basic algebra yields (12c) ■

**Proposition V.3.** *With $p_m > 0$, and $\lambda_0 = \mu_m = 0$, (11) imply*

$$h_m = c/B(x_m^*;\mathbf{a_m}) \tag{13}$$

*Proof:* With $\mu_m = 0$, (12b) yields $B'(x_m;\mathbf{a_m}) = 0 \implies x_m = x_m^*$ (Fig. 2), which replaced into (12a) yields (13). ■

*Remark* V.1. Because $c$ and $h_m$ are exogenous, and $B(x_m^*;\mathbf{a_m})$ is a property of an S-curve associated with $\mathbf{a_m}$, (13) can only hold by sheer coincidence. Thus, Prop. V.3 means that if there is unused power at the optimum ($\lambda_0 = 0$), then over every used channel the symbol rate must equal its maximal value.

**Theorem V.1.** *If $(p_1, \cdots, p_M)$ is a (local) optimiser for Problem (10), and $p_m > 0$ with $\mu_m > 0$, then there is a non-negative $\lambda_0$ such that*

$$\frac{H_m}{N_0} \bar{b}_m f'(x_m;\mathbf{a_m}) = c + \lambda_0 \tag{14}$$

*Proof:*

$p_m > 0 \implies x_m > 0$. With $\mu_m > 0$, (11d) implies that $x_m = h_m p_m / (\rho^* \hat{R}_m)$, which, together with (12c), implies

$$h_m\left(B(x_m;\mathbf{a_m}) + x_m B'(x_m;\mathbf{a_m})\right) = c + \lambda_0 \tag{15a}$$

$$\text{But, } B(x_m;\mathbf{a_m}) + x_m B'(x_m;\mathbf{a_m}) \equiv \frac{d}{dx}\left\{x B(x;\mathbf{a_m})\right\} \tag{15b}$$

And, by (8), $x B(x;\mathbf{a_m}) \equiv \bar{b}_m f(x;\mathbf{a_m})/\rho^*$, thus

$$\frac{d}{dx}\left\{x B(x;\mathbf{a_m})\right\} \equiv \frac{1}{\rho^*} \bar{b}_m f'(x_m;\mathbf{a_m}) \tag{15c}$$

$\therefore$ By (15c) and (7), (15a) is equivalent to (14).

■

Let $\tilde{x}_m$ denote the inflexion point of $f(\cdot;\mathbf{a_m})$, and

$$\delta_m^* := f'(\tilde{x}_m;\mathbf{a_m}) \tag{16}$$

i. e., $\delta_m^*$ is the largest value of the derivative of $f(\cdot;\mathbf{a_m})$.

**Proposition V.4.** *Equation (14) has at most 2 solutions, $x_{11}$, $x_{12}$, and they satisfy $x_{11} \leq \tilde{x}_m \leq x_{12}$.*

*Proof:* Because $f$ satisfies Definition A.1, $f'$ satisfies Definition A.2; i.e., it is strictly increasing over $[0, \tilde{x}_m)$ and strictly decreasing over $(\tilde{x}_m, \infty)$ (see Fig. 1), and so is the function on the left side of (14), whose highest possible value is $H_m \bar{b}_m \delta^*/N_0$. The thesis follows from Lemma A.1. ∎

**Proposition V.5.** *If $H_m < cN_0/(\delta_m^* \bar{b}_m)$ then any pair of the form $p_m > 0$, $R_m = \hat{R}_m$ is non-optimal.*

*Proof:* Sketch: (14) is necessary but with $H_m < cN_0/(\delta_m^* \bar{b}_m)$, *no* $\lambda_0 \geq 0$ can satisfy (14). ∎

**Conjecture V.1.** *A solution to (14) that is less than the inflexion point of $f'(\cdot;\mathbf{a_m})$ is not a maximiser.*

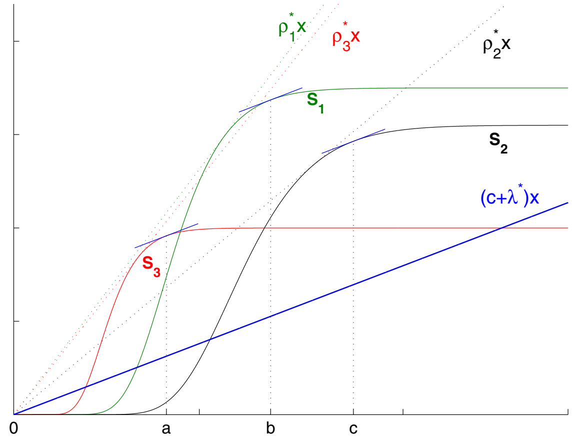
*Remark* V.2. Conjecture V.1 can be confirmed through the second-order optimising conditions. Notice that the right side of (14) is independent of $H_m$. Thus, after an increase in $H_m$, in order for the equality to remain satisfied, the value of the derivative on the left side of (14) need to decrease. Because of the "bell" shape of the derivative, if the solution to the left of $\tilde{x}_m$ is chosen as the optimiser ($x_{11}$), one would move to the left of $x_{11}$ in order to get the lower derivative value that satisfies (14). But this is counter-intuitive, because it would reduce the SNR of a channel whose quality has actually improved.

*Remark* V.3. An economic interpretation of (14) provides useful insight. Suppose that a "selfish agent" is assigned to each channel, and asked to choose whichever power it wants at a unit cost of $c + \lambda_0$. The agent chooses the quantity that maximises its own utility ("benefit minus cost"):

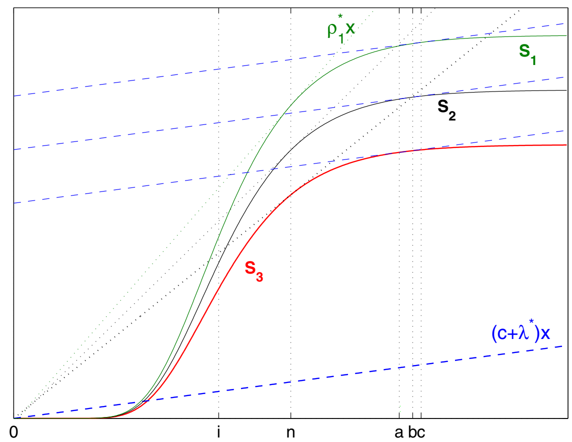
$$\bar{b}_m \hat{R} f(x;\mathbf{a_m}) - (c+\lambda_0) x \hat{R}/(H_m/N_0) \tag{17}$$

or equivalently $S_m(x) - (c+\lambda_0)x$ with $S_m(x) = (H_m/N_0)\bar{b}_m f(x;\mathbf{a_m})$ (the constant $\hat{R}$ has no impact). This leads to $S'(x) - (c+\lambda_0) = 0$ or $(H_m/N_0)\bar{b}_m f'(x;\mathbf{a_m}) = c + \lambda_0$, which is identical to (14). Each agent then obtains the largest solution to that equation, $x_m$, from which it gets a power level: $p_m = x_m \hat{R}/(H_m/N_0)$. However, since each agent acts independently, for an arbitrary $\lambda_0$, their total "demand" may very well exceed the available "supply" (power constraint), or leave unused power (which would require $\lambda_0 = 0$ by (11c)). This suggests an approach to find the right $\lambda_0$.

*Remark* V.4. Fig. 3a shows the graphs for 3 $S_m(x) = (H_m/N_0)\bar{b}_m \hat{R} f(x;\mathbf{a_m})$. For a given $c + \lambda_0$ each "agent" finds the SNR value that makes $S'_m(x) = c + \lambda_0$; i. e., a value at which a tangent to the curve is parallel to the price line $(c+\lambda_0)x$. One can find the right $\lambda_0$ by "sweeping" a price line from vertical ($\lambda_0 = \infty$) towards the horizon. If $c + \lambda_0$ exceeds the slope of the steepest tangenu ($\rho_1^*$ herein) each agent chooses 0 (thus $c + \lambda_0$ can be started at $\rho_1^*$). With $c + \lambda_0 = \rho_1^*$, only agent 1 buys whichever power produces $x_1 = x_1^*$ (exactly at the "genu" of $S_1$). This power level may be less than the constraint, suggesting that $c + \lambda_0$ be made even lower, which would make agent 1 buy a larger quantity, and if $c + \lambda_0 \leq \rho_3^*$ also make agent 3 buy an appropriate level, and so on. Eventually, the process ends with $\lambda_0 = \lambda^*$. At that level, the chosen power levels (corresponding SNR's $x_1 = b$, $x_2 = c$ and $x_3 = a$) add up to the power constraint. The order in which channel agents "buy" is determined by the slopes of the tangenus of the S-curves.




(a) Dissimilar link configurations




(b) Common link configuration

Figure 3: Finding the optimal $\lambda_0$

In Fig. 3b, the link configuration is common, thus the S-curves are multiples of each others, and each curve's genus occurs at the common SNR value, $n$. In this case, the first agent to "buy" is necessarily the one corresponding to the "best" channel ("taller S-curve"), the second to buy corresponds to the second "best", and so on. The chosen SNR values are $x_1 = c$, $x_2 = b$, $x_3 = a$ with $a < b < c$.

## VI. Summary/Discussion

We have analysed the general problem of allocating a power budget over several sub-channels to maximise a general throughput function while considering the cost of power. Some precise answers have arisen.

- By Proposition IV.3, if the total available power is "low" then all power should go to the best sub-channel, and the link configuration is identified in Fig. 2.
- By Proposition IV.1, if a suitably normalised sub-channel gain is less than the normalised power cost then the channel can be discarded from the outset.
- By Proposition IV.2, at the optimum, at most one sub-channel, say $n$, will operate at a symbol-rate below the rate constraint ($R_n < \hat{R}_n$), with the link configuration and operating point shown in Fig. 2.
- By Proposition V.3, if at the optimum the available power is not fully exhausted (which is possible because power is costly), then all active sub-channels operate at their maximal symbol rate.
- The main result, Theorem V.1, shows that the SNR of a terminal that, at the optimum, operates at its maximal symbol rate is one (the larger) of the at most 2 solutions of a single-variable equation (14).
- The analysis leads to a general solution for the case where power is not "low":

Remark V.4 describes graphically an algorithm that leads to the solution, whose computer implementation seems straightforward. We assumed that each used sub-channel can reach the KKT SNR value (14) while operating at the maximal symbol rate. More likely, the power remaining for the "last" activated sub-channel will be insufficient for it to reach the KKT SNR at maximal symbol rate. One then adjust the symbol rate so that with the remaining power it can operate at an efficient SNR (recall Fig. 2). Future versions of this work may provide second-order analysis, and the results of numerical experiments.

**Conjecture VI.1.** *At the optimum, $M^* \leq M$ channels operate at the maximal symbol rate, each with respective SNR obtained as the largest of the 2 solutions of (14). Additionally, at most one channel operates at the optimal SNR $x^*$ with the ideal link configuration $\mathbf{a}^*$ (Fig. 2).*

## ACKNOWLEDGEMENT

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## APPENDIX

**Definition A.1.** $\mathcal{S} : \Re_+ \to [0, Y]$, is an S-curve with unique inflexion at $x_f$ if (i) $\mathcal{S}(0) = 0$, $\mathcal{S}$ is (ii) continuously differentiable, (iii) strictly increasing, (iv) convex over $[0, x_f)$ and concave over $(x_f, \infty)$, and (v) surjective (see Fig. 1).

**Definition A.2.** A function $h : \Re_+ \to [0, Y]$ is single-peaked over $\Re_+$ if $h$ is continuous, surjective and has a global maximum at $X \in (0, \infty)$ (that is, $h(X) = Y$ , $0 \leq x_1 < x_2 \leq X \implies h(x_2) > h(x_1)$ and $X \leq x_1 < x_2 \implies h(x_2) < h(x_1)$).

**Lemma A.1.** *Let $h : [d, e] \to \Re_+$ be a continuous strictly quasi-concave function $h$ with maximal value $Y$ at $X$ where $d < X < e$, that is, $d \leq x_1 < x_2 \leq X \implies h(x_2) > h(x_1)$*

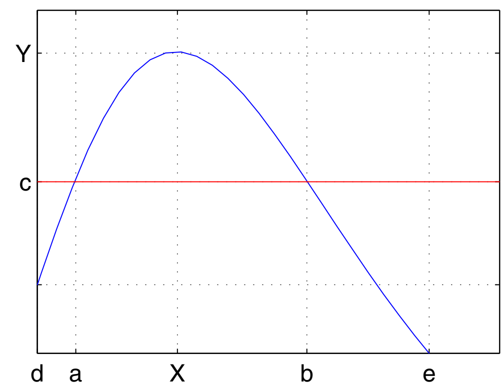


Figure 4: Graph of a single-peaked function $h$ with maximal value $Y$ at $X$ .

*and $X \leq x_1 < x_2 \leq e \implies h(x_1) > h(x_2)$ (Fig. 4). Suppose $h(d) \geq h(e)$. Let $\mathcal{S} := \{x : h(x) = c\}$ (the set of solutions to $h(x) = c$).*

$$\mathcal{S} = \begin{cases} \emptyset & \text{if } c > Y \text{ or } c < h(e) \\ \{X\} & \text{if } c = Y \\ \{a, b\} \text{ with } a < X < b & \text{if } h(d) < c < Y \\ \{b\} \text{ with } X < b & h(e) < c < h(d) \end{cases}$$

*Furthermore, $a < x < b \implies h(x) > c$ .*

*Proof:* Omitted. ∎

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