On the Generation of Rayleigh Fading Processes with Accurate Statistical Properties

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***Abstract*— This paper proposes an algorithm for the generation of Rayleigh fading processes. This algorithm is based on the sum-of-sinusoids method proposed by Clarke in [1] and modified by Aulin in [2]. Clarke's model is taken as a reference and it is complemented considering [3], obtaining as a result a well described algorithm for the generation of Rayleigh fading processes.**

***Index Terms*— Sum-of-sinusoids method, simulation of Rayleigh fading, algorithm for generation of short-term fading.**

I. Introduction

Short-term fading refers to the changes in signal amplitude and phase that can be experienced as a result of small changes in the spatial separation between transmitter and receiver. This kind of fading is also referred to as multi-path fading. The two basic mechanisms that cause multi-path propagation in mobile radio systems are reflection and scattering [4]. They lead to a situation in which the received signal is made up of a number of waves, whose angles of arrival occur randomly for different positions of the receiver. Furthermore, their phases are completely random such that they are uniformly distributed over $[-\pi,\pi)$ [1].

In this paper the model of sum-of-sinusoids for generation of Rayleigh fading first proposed by Clarke in [1] will be taken as a reference. This model has also been taken as a reference by other authors who have characterized multi-path propagation and its statistical properties. In [2], Aulin modified the original model considering that the assumption of waves traveling horizontally is not true in an urban environment. However, the model is not frequently used for the generation of fading processes, because of its complexity and because the results obtained with Clarke's model are satisfactory for simulation purposes. Moreover, the statistical properties obtained with this model are correct. In the present paper, the theoretical approximations as shown in [2] are used and a simplification is done in order to come to the original model.

As already mentioned, other authors have investigated the generation of Rayleigh fading and its statistical properties. In [3], Clarke's model is reconsidered and a redefinition of its random variables is done. The paper considers the approach investigated by Pop and Beaulieu in [5] and includes a review of the deterministic model of Jakes, [6], which has been widely used and studied over the time. In [5], Pop and Beaulieu demonstrated that the probability density function (pdf) of the signal envelope produced by Jakes' simulator averaged across the fading ensemble is a function of time. They conclude that its ensemble average is not stationary, not even wide sense stationary. This information is also considered by Xiao, Zheng and Beaulieu in [7]. In the present paper this ensemble average approximation is considered and applied in an algorithm with a new set of generating equations based in the Aulin model introduced in [2].

It is also worth to mention the so called *deterministic* simulation models for the generation of Rayleigh fading. These methods are based on the development of Jakes and they have been exhaustively investigated by Pätzold *et al.* in [8] and [9], among others. The authors use in their papers the methods explained in [13]. This kind of generators is not considered in the present paper due to the reasons explained in [5], i.e., sums of fixed amplitude, random-phase sinusoids are not ergodic and stationary, thus, the time averages may or may not equal to the stochastic ones. These methods are also used in [10]-[12] for the simulation of uncorrelated and cross-correlated fading channels. In those papers the objective of the generation of Rayleigh fading channels characteristics is achieved, but the higher order statistics of the generated processes are not mentioned.

In the procedure for Rayleigh fading generation proposed in the present paper, the original model in [1], and modified by [2], is used and complemented considering [3]. It will be shown, that the statistics of first and higher orders agree with theoretical results for the new model. The most important contribution of the paper is that the statistical propierties of the new model are closer to the reference than in previous papers. The remainder of this paper is organized as follows, Section II describes the model in [2] and its statistical properties. Section III presents the algorithm to be followed for the generation of the Rayleigh fading processes and the statistical properties of the processes generated with it. In Section IV the processes generated with this algorithm are evaluated and finally in Section V some conclusions are drawn.

II. Aulin's Model

A. Mathematical description

Being N the number of superimposed waves (signal paths) and the n_{th} incoming wave being characterized by spatial angles of arrival α_n to the xz plane and β_n to the xy plane. The n_{th} incoming wave also has an amplitude c_n and a phase shift ϕ_n. Moreover,

$$E\{c_n^2\} = \frac{E_0}{N} \tag{1}$$

Denoting the angular carrier frequency by ω_c, the resulting field can be written as in [2]

$$E(t) = \sum_{n=1}^{N} E_n(t) \tag{2}$$

where

$$E_n(t) = c_n \cos[\omega_c t + \frac{2\pi}{\lambda}(x_0 \cos\alpha_n \cos\beta_n + y_0 \sin\alpha_n \cos\beta_n + z_0 \sin\beta_n) + \phi_n] \quad (3)$$

Being α_n, β_n and ϕ_n mutually independent and uniformly distributed over $[-\pi, \pi)$. In Eq. 3, the minus sign of Eq. 5 in Aulin's paper is corrected because of the consideration that it is not consistent with the results given by Aulin in his Eqs. 6, 7 and 8.

In a point (x_0, y_0, z_0) and with a mobile station traveling with a velocity v in a direction with angle γ to the xz plane,

$$E_n(t) = c_n \cos[\omega_c t + \frac{2\pi}{\lambda}(vt \cos\gamma \cos\alpha_n \cos\beta_n + vt \sin\gamma \sin\alpha_n \cos\beta_n + z_0 \sin\beta_n) + \phi_n] \quad (4)$$

And simplifying for $\beta_n = 0$, as assumed by Clarke

$$E_n(t) = c_n \cos\left[\omega_c t + \frac{2\pi}{\lambda} vt \cos(\gamma - \alpha_n) + \phi_n\right] \quad (5)$$

Therefore $E(t)$ can be expressed as

$$E(t) = T_c(t)\cos(\omega_c t) - T_s(t)\sin(\omega_c t) \quad (6)$$

where

$$T_c(t) = \sum_{n=1}^{N} c_n \cos(\omega_d t \cos(\gamma - \alpha_n) + \phi_n)$$
$$T_s(t) = \sum_{n=1}^{N} c_n \sin(\omega_d t \cos(\gamma - \alpha_n) + \phi_n) \quad (7)$$

Here $\omega_d = 2\pi v/\lambda$ is the maximum radian Doppler frequency and the envelope and phase of the complex signal $T(t) = T_c(t) + jT_s(t)$ are given by

$$|T(t)| = \sqrt{T_c(t)^2 + T_s(t)^2} \quad (8)$$

$$\theta(t) = \arctan\left[\frac{T_s(t)}{T_c(t)}\right] \quad (9)$$

B. Statistical Properties

According to the central limit theorem, the processes $T_c(t)$ and $T_s(t)$ are approximately Gaussian for large N. Thus, for the statistics, these processes are regarded as purely Gaussian.

The statistical properties of the Aulin's model can be consulted in [2] and [15] for the autocorrelations and cross-correlations of $T_c(t)$, $T_s(t)$ and $|T(t)|^2$, summarizing

$$R_{T_cT_c}(\tau) = J_0(\omega_d \tau) \quad (10a)$$
$$R_{T_sT_s}(\tau) = J_0(\omega_d \tau) \quad (10b)$$
$$R_{T_cT_s}(\tau) = 0 \quad (10c)$$
$$R_{T_sT_c}(\tau) = 0 \quad (10d)$$
$$R_{TT}(\tau) = 2J_0(\omega_d \tau) \quad (10e)$$
$$R_{|T|^2|T|^2}(\tau) = 4 + 4J_0^2(\omega_d \tau) \quad (10f)$$

Where $E_0 = 2$ is taken for normalization purposes.

III. Algorithm for the generation of Rayleigh fading processes

A. Algorithm

In Figure 1 the algorithm for the generation of Rayleigh fading processes is shown. It is composed of three steps. First of all the definitions and initializations of the variables are carried out. Here, *size(A)=b* assigns a size of b to the vector A and *init(A,b)* initializes every position of vector A with a value of b. Furthermore, $\sim U[-\pi, \pi)$ generates a random number uniformly distributed over $[-\pi, \pi)$. In the second step a loop is executed. In this loop an ensemble average over N_{tr} trials is done. This is carried out with the goal of achieving stationarity, as mentioned in Section I. After the summation of the N_{tr} trials, the resulting value is divided by $\sqrt{N_{tr}}$ for normalization purposes in the correlations of the process. In every trial of the ensemble average, T_{rc} and T_{rs} processes are generated as defined in Eqs. 7. Here, $\alpha_n = \frac{2\pi n - \pi + \eta[tr]}{N} - \pi$ for improving the performance of the algorithm, since the generation of an uniformly distributed number is not needed in every step of the summation. Moreover, this value fulfils the initial requirements of the mathematical model. As can be seen η, γ and ϕ_n are independent and uniformly distributed random variables over $[-\pi, \pi)$.

The third and last step is the calculation of the discrete ensemble average process $T_r(t)$.

Step 1: Definitions and initializations
  N_{tr} := *Number of trials*
  N := *Number superimposed waves (signal paths)*
  v := *Mobile velocity in* km/h, $v = v/3600$
  c := *Speed of light in* km/s, f_c := *Carrier frequency*
  $\lambda = c/f_c$, $f_d = v/\lambda$, $\omega_d = 2\pi f_d$, $T_d = 0.025/f_d$
  t_m := *Maximal evaluated time (a factor of* T_d*)*
  $k_m = t_m/T_d$
  size(T_{rc}), *size*$(T_{rs}) = k_m + 1$
  init$(T_{rc}, 0)$, *init*$(T_{rs}, 0)$
Step 2: Calculation of the Processes
  for tr=1:N_{tr}
    $\gamma[tr] \sim U[-\pi, \pi)$
    $\eta[tr] \sim U[-\pi, \pi)$
    for k=1:$k_m + 1$
      $T_c = 0$
      $T_s = 0$
      for n=1:N
        if k==1
          $\phi[n] \sim U[-\pi, \pi)$
        $\alpha_n = (2\pi n - \pi + \eta[tr])/N - \pi$
        $cos\gamma\alpha_n = \cos(\gamma[tr] - \alpha_n)$
        $T_c \mathrel{+}= \cos(\phi[n] + \omega_d T_d (k-1) \cdot cos\gamma\alpha_n)$
        $T_s \mathrel{+}= \sin(\phi(n) + \omega_d T_d (k-1) \cdot cos\gamma\alpha_n)$
      $T_{rc}[k] \mathrel{+}= \sqrt{2/N} T_c$
      $T_{rs}[k] \mathrel{+}= \sqrt{2/N} T_s$
  $T_{rc} = T_{rc}/\sqrt{N_{tr}}$
  $T_{rs} = T_{rs}/\sqrt{N_{tr}}$
Step 3: T(t) Process
  $T_r(t) = T_{rc} + jT_{rs}$

Fig. 1. Algorithm for the generation of Rayleigh fading processes

B. Statistical Properties

The statistical properties for the process, with the chosen value of α_n are the same as in Eq. 10. Therefore, the chosen implementation fulfils the requirements of the theoretical

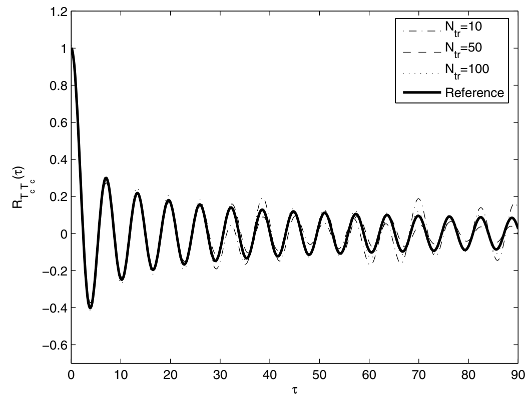

Fig. 2. Autocorrelation of the in-quadrature component

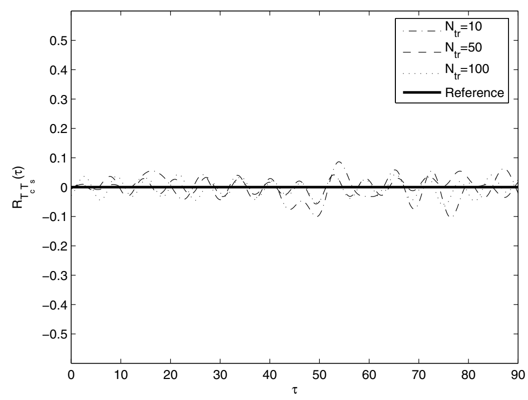

Fig. 3. Cross correlation of the in-quadrature and in-phase components

model. The demostration for the autocorrelation of the in-quadrature component can be seen in Appendix I. The proof of the remaining equations can be done in an analogous way (they are not going to be showed here due to space restrictions).

IV. Evaluation of the Statistical Properties of the Generated Process

In Figs. 2-8 both, the statistics of the generated process and the theoretical values of these statistics are shown for different values of number of trials. In the figures, the theoretical value is named Reference and is defined as in Eqs. 10. N_{tr} corresponds to the number of trials used during the simulations. Although an optimisation of the values of N and N_{tr} was carried out, it is not possible to show the process because of space restrictions. The optimal values foud were $N = 15$ and $N_{tr} = 10$, this explanation will be subject of a posterior paper. Even though the optimal value of $N_{tr} = 10$, values of number of trials of 50 and 100 were also simulated for comparison purposes, i.e., to be able to compare with the results in [3]. As expected, by increasing the value of number of trials, the approximation to the theoretical value increases. The number of superimposed waves used in the algorithm for the generation of the figures is $N = 15$, which is a low value compared with simulations in previous papers. In these papers the authors normally redefine a value of $N_0 = \frac{1}{2}(\frac{N}{2} - 1)$ as in [6] or $M = \frac{N}{4}$ as in [3]. In the latter, the authors used a value of $M = 8$, i.e., $N = 32$, for the sum-of-sinusoids in their simulations. It is important to consider, that although a sums of $N = 15$ sinusoids are more time and resource consuming than a sums of $M = 8$ sinusoids, in the previous models every component of the sum also had to be multiplied either by a cosine or by a sine depending on the component of $T(t)$ that is to be found. Therefore, the performance of the algorithms is comparable.

A. Evaluation of Correlation Statistics

Figs. 2-6 show the statistics of the different simulated processes for every value of N_{tr} mentioned in Section IV. It can be observed that the statistics of the simulated processes are very close to the theoretical values, this remains valid for a value of N_{tr} as small as 10. Even Fig. 6, which shows the autocorrelation of the squared envelope and contains fourth-order statistics, [3], is very approximated to the reference. It can be seen that the model fulfils the statistical properties

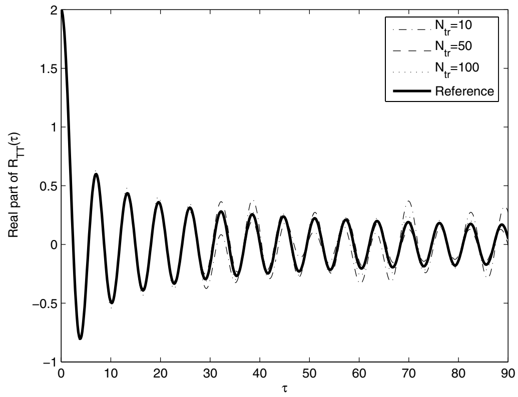

Fig. 4. Real part of the autocorrelation of the complex fading

of the Rayleigh fading processes. Furthermore, by comparing these results with results of previous models, it becomes clear that the statistics of the simulated processes applying the new set of equations as in Figs. 2-4 and Fig. 6 are nearer to the reference model than in previous models for the same number of trials. On the other hand, the statistical properties in Fig. 5 are closer to the reference in the models developed before, however, this is hardly noticeable.

B. Evaluation of the pdf of the envelope and the phase

As already known, the pdfs of the envelope and phase are Rayleigh and uniformly distributed respectively, they are given by

$$f_{|T|}(x) = xe^{-x^2/2}, \qquad x \geq 0 \tag{11a}$$

$$f_{\Theta_T}(\theta_T) = \frac{1}{2\pi}, \qquad \theta_T \in [-\pi, \pi) \tag{11b}$$

Figs. 7 and 8 show the pdf of the fading envelope and phase of the simulated processes. As can be seen, they overlap almost totally the theoretical values. In the pdf of the phase there is a significant improvement compared to figures in previous papers.

V. Conclusion

In this paper, an algorithm for the generation of Rayleigh fading processes with accurate statistical properties was proposed (Fig. 1). In this algorithm an ensemble average over a

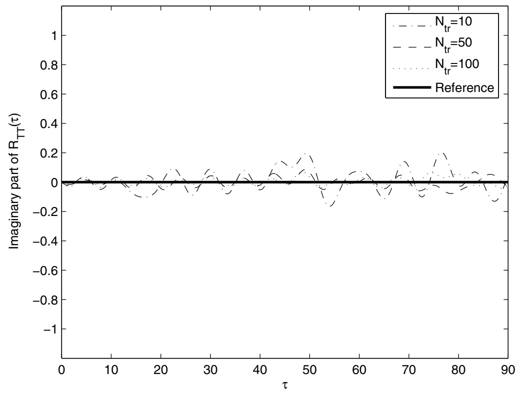

Fig. 5. Imaginary part of the autocorrelation of the complex fading

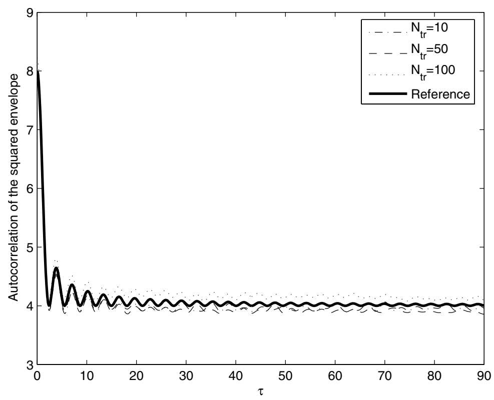

Fig. 6. Autocorrelation of the squared envelope of fading

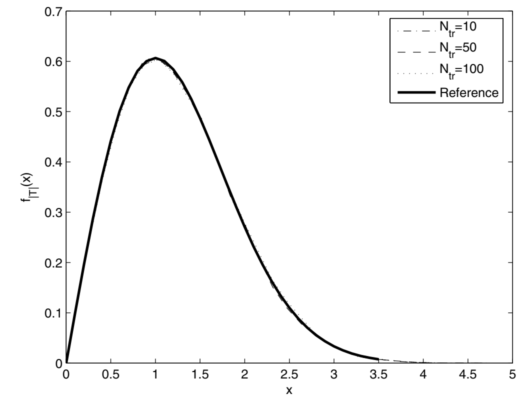

Fig. 7. PDF of the fading envelope

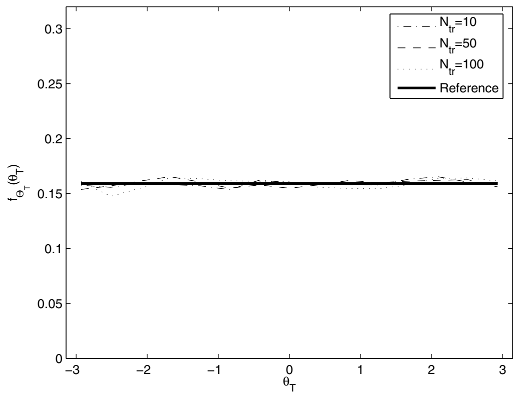

Fig. 8. PDF of the fading phase

defined number of trials is carried out as in [3] and [7] for meeting statistical properties, which were not fulfiled before, as mentioned in [5]. As can be expected, with an increasing number of trials, the accuracy of the statistical properties of the process rises. But this accuracy does not only depend on the number of trials but also on the number of incoming waves used during the simulations. For the case of $N = 15$ sinusoids studied in this paper and for a value of number of trials N_{tr} as low as 10, the generated random variables meet the expected statistical properties, more important, they are closer to the reference value than in previous models. Hence, the generation of processes with accurate statistical properties is better achieved with the new set of equations proposed in the paper and the corresponding algorithm. Moreover, the use of computational resources of the algorithm is comparable to previous models due to the reasons explained in Section IV.

APPENDIX I
AUTOCORRELATION OF THE IN-QUADRATURE COMPONENT

$$\begin{aligned}R_{T_cT_c}(\tau)=&E\{T_c(t)T_c(t+\tau)\}\\=&E\{\sum_{n=1}^{N} c_n\cos\left(\omega_d t\cos(\gamma-\alpha_n)+\phi_n\right)\\&\sum_{m=1}^{N} c_m\cos\left(\omega_d(t+\tau)\cos(\gamma-\alpha_m)+\phi_m\right)\}\end{aligned}$$

$$\begin{aligned}=&\frac{2}{N}\frac{1}{2}\sum_{n,m=1}^{N} E\{\cos\left[\omega_d t\cos\left(\gamma-\alpha_n\right)\right.\\&\left.+\omega_d(t+\tau)\cos\left(\gamma-\alpha_m\right)+\phi_n+\phi_m\right]\}\\&+\frac{2}{N}\frac{1}{2}\sum_{n,m=1}^{N} E\{\cos\left[\omega_d t\cos\left(\gamma-\alpha_n\right)\right.\\&\left.-\omega_d(t+\tau)\cos\left(\gamma-\alpha_m\right)+\phi_n-\phi_m\right]\}\end{aligned}$$

Where the stochastic variables δ_1 and δ_2 can be defined as in Appendix I of [2]

$$\begin{aligned}\delta_1 &= (\phi_n+\phi_m)_{mod\ 2}\\ \delta_2 &= (\phi_n-\phi_m)_{mod\ 2}\end{aligned}$$

These two stochastic variables are uniformly distributed over $[-\pi,\pi)$ for all m and n except δ_2 when $m = n$. When $m = n$, δ_2 equals zero with probability 1, thus, taking the expectation over ϕ_m and ϕ_n,

$$R_{T_cT_c}(\tau) = \frac{1}{N}\sum_{n=1}^{N} E\{\cos\left[-\omega_d\tau\cos\left(\gamma-\alpha_n\right)\right]\}$$

And with the value of α_n given in the algorithm, i.e., $\alpha_n = \frac{2\pi n-\pi+\eta}{N} - \pi$, which is calculated in order to have the desired integration limits,

$$\begin{aligned}R_{T_cT_c}(\tau)=&\frac{1}{N}\sum_{n=1}^{N} E\{\cos\left[\omega_d\tau\cos\left(\gamma-\alpha_n\right)\right]\}\\=&\frac{1}{N}\sum_{n=1}^{N}\int_{-\pi}^{\pi}\cos\left[\omega_d\tau\cos\left(\frac{2\pi n-\pi+\eta}{N}-\pi-\gamma\right)\right]\frac{d\eta}{2\pi}\end{aligned}$$

$$=\frac{1}{2\pi}\sum_{n=1}^{N}\int_{\frac{2\pi(n-1)}{N}-\pi-\gamma}^{\frac{2\pi n}{N}-\pi-\gamma}\cos\left[\omega_d\tau\cos\left(u\right)\right]du$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left[\omega_d\tau\cos\left(u\right)\right]du$$
$$=J_0(\omega_d\tau)$$

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